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# EIGENVALUE ESTIMATES FOR OPERATORS WITH FINITELY MANY NEGATIVE SQUARES

JUSSI BEHRNDT, ROLAND MÖWS, AND CARSTEN TRUNK

*Dedicated with great pleasure to our friend Petru Cojuhari on the occasion of his 65th birthday*

ABSTRACT. Let  $A$  and  $B$  be selfadjoint operators in a Krein space. Assume that the resolvent difference of  $A$  and  $B$  is of rank one and that the spectrum of  $A$  consists in some interval  $I$  of isolated eigenvalues only. In the case that  $A$  is an operator with finitely many negative squares we prove sharp estimates on the number of eigenvalues of  $B$  in the interval  $I$ . The general results are applied to singular indefinite Sturm-Liouville problems.

## 1. INTRODUCTION

Let  $A$  and  $B$  be bounded or unbounded selfadjoint operators in a Krein space  $(\mathcal{K}, [\cdot, \cdot])$  such that  $\rho(A) \cap \rho(B) \neq \emptyset$  and assume that the resolvent difference of  $A$  and  $B$  is a rank one operator,

$$\dim(\operatorname{ran}((A - \lambda_0)^{-1} - (B - \lambda_0)^{-1})) = 1$$

for some, and hence for all,  $\lambda_0 \in \rho(A) \cap \rho(B)$ . The main objective of this note is to provide sharp bounds on the number of distinct isolated eigenvalues of  $B$  in an open interval  $I \subset \mathbb{R}$  in terms of the number of distinct isolated eigenvalues of  $A$  in  $I$ . In our considerations also eigenvalues of infinite geometric or algebraic multiplicity are allowed and hence multiplicities are not counted.

In the indefinite setting of Krein spaces additional assumptions on the operators  $A$  and  $B$  have to be imposed in order to obtain meaningful bounds and estimates. In particular, it may happen that an open interval  $I \cap \mathbb{R}$  is contained in  $\rho(A)$  but the set  $\sigma_p(B) \cap I$  is infinite, or it may even happen that  $I$  is in the interior of  $\sigma(B)$ , see, e.g., [24, §5, Remark 2]. In the recent work [3] the special case of a selfadjoint operator  $A$  which is nonnegative with respect to the indefinite inner product  $[\cdot, \cdot]$  was considered; we refer the reader there for more details and further references.

In the present paper we go beyond the nonnegative case and consider a selfadjoint operator  $A$  with finitely many negative squares  $\kappa_A$ , that is, the form  $[A\cdot, \cdot]$  has negative values on a  $\kappa_A$ -dimensional subspace of  $\operatorname{dom} A$ , but there is no  $\kappa_A + 1$ -dimensional subspace with this property. We then employ typical techniques from perturbation theory to study interlacing properties of eigenvalues of  $A$  and  $B$  in a gap of the essential spectrum. In some sense we are able to control the number of eigenvalues of  $A$  and  $B$  which may destroy natural interlacing. Obviously, those eigenvalues are of special interest. We do this in terms of a local quantity related to the number of negative squares of  $A$  and  $B$ , respectively. This analysis leads to upper and lower bounds on the number of distinct eigenvalues of  $B$  in a gap of the essential spectrum of  $A$  in our first main result Theorem 3.1. It is remarkable that all these bounds are sharp; this is our second main result formulated as Theorem 4.1.

Finally, in the last section, the abstract results are illustrated for a typical Sturm-Liouville eigenvalue problem with an indefinite weight function which gives rise a selfadjoint operator with finitely many negative squares in a weighted  $L^2$ -Krein space.

## 2. OPERATORS WITH FINITELY MANY NEGATIVE SQUARES AND RANK ONE PERTURBATIONS

A complex linear space  $\mathcal{K}$  with a nondegenerate hermitian sesquilinear form  $[\cdot, \cdot]$  is called a *Krein space* if there exists a decomposition

$$\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$$

such that the subspaces  $(\mathcal{K}_\pm, \pm[\cdot, \cdot])$  are Hilbert spaces and orthogonal to each other with respect to  $[\cdot, \cdot]$ . An element  $x$  in the Krein space  $(\mathcal{K}, [\cdot, \cdot])$  is *positive* (*negative*, *neutral*) if  $[x, x] > 0$  ( $[x, x] < 0$ ,  $[x, x] = 0$ , respectively). For further information on Krein spaces we refer to the monographs [1] and [10].

For a densely defined linear operator  $A$  in the Krein space  $(\mathcal{K}, [\cdot, \cdot])$  the adjoint with respect to the indefinite inner product  $[\cdot, \cdot]$  is denoted by  $A^+$ . The operator  $A$  is called *selfadjoint* if  $A = A^+$  and *symmetric* if  $A \subset A^+$ . We denote the point spectrum by  $\sigma_p(A)$ , the spectrum by  $\sigma(A)$  and the resolvent set by  $\rho(A)$ . The *root subspace*  $\mathcal{L}_\lambda(A)$  of  $A$  at  $\lambda$  is the collection of all Jordan chains,  $\mathcal{L}_\lambda(A) := \{x \in \ker(A - \lambda)^j : j \in \mathbb{N}\}$ . A real isolated eigenvalue  $\lambda$  of  $A$  is called of *positive* (*negative*) *type* if all its corresponding eigenvectors are positive (negative, respectively). In this case we write  $\lambda \in \sigma_{++}(A)$  ( $\lambda \in \sigma_{--}(A)$ , respectively). An isolated eigenvalue  $\lambda$  of  $A$  which is not of positive neither of negative type is called a *critical point* of  $A$  and we write  $\lambda \in c(A)$ . Observe that for an isolated eigenvalue of positive or negative type there is no Jordan chain of length greater than one, that is,  $\mathcal{L}_\lambda(A) = \ker(A - \lambda)$ .

A selfadjoint operator  $A$  in the Krein space  $(\mathcal{K}, [\cdot, \cdot])$  with  $\rho(A) \neq \emptyset$  has  $\kappa_A$  *negative squares* if for some  $\kappa_A \in \mathbb{N}$  the hermitian form  $\langle \cdot, \cdot \rangle$  on  $\text{dom } A$ , defined by

$$\langle f, g \rangle := [Af, g], \quad f, g \in \text{dom } A,$$

has  $\kappa_A$  negative squares, that is, there exists a  $\kappa_A$ -dimensional subspace  $\mathcal{M}$  in  $\text{dom } A$  such that  $\langle v, v \rangle < 0$  if  $v \in \mathcal{M}$ ,  $v \neq 0$ , but no  $\kappa_A + 1$  dimensional subspace with this property. Selfadjoint operators with finitely many negative squares belong to the class of definitizable operators introduced and comprehensively studied by H. Langer in [22, 23]. We recall some well-known spectral properties of operators with finitely many negative squares. The statements in Theorem 2.1 below can be found in [22, 23], see also [5, Theorem 3.1].

**Theorem 2.1.** *Let  $A$  be a selfadjoint operator in the Krein space  $(\mathcal{K}, [\cdot, \cdot])$  and assume that  $A$  has  $\kappa_A$  negative squares. Then the nonreal spectrum of  $A$  consists of at most  $\kappa_A$  pairs  $\{\mu_i, \bar{\mu}_i\}$ ,  $\text{Im } \mu_i > 0$ , of eigenvalues with finite-dimensional root subspaces. If  $\{\kappa_-(\lambda), \kappa_0(\lambda), \kappa_+(\lambda)\}$  denotes the signature of the root subspace  $\mathcal{L}_\lambda(A)$  then*

$$\sum_{\lambda \in \sigma_p(A) \cap (0, \infty)} (\kappa_-(\lambda) + \kappa_0(\lambda)) + \sum_{\lambda \in \sigma_p(A) \cap (-\infty, 0)} (\kappa_+(\lambda) + \kappa_0(\lambda)) + \sum_i \kappa_0(\mu_i) \leq \kappa_A$$

*and if  $0 \notin \sigma_p(A)$  then equality holds. Moreover, there are at most  $\kappa_A$  different real nonzero eigenvalues of  $A$  with corresponding Jordan chains of length greater than one.*

Let  $A$  and  $B$  be selfadjoint operators in the Krein space  $(\mathcal{K}, [\cdot, \cdot])$ , let  $\rho(A) \cap \rho(B) \neq \emptyset$  and assume that

$$(2.1) \quad \dim \text{ran}((A - \lambda_0)^{-1} - (B - \lambda_0)^{-1}) = 1$$

for some (and hence for all)  $\lambda_0 \in \rho(A) \cap \rho(B)$ . It is not difficult to see that if  $A$  and  $B$  satisfy Assumption (I) and  $A$  has  $\kappa_A$  negative squares then it follows that also the operator  $B$  has  $\kappa_B \geq 0$  negative squares, where

$$(2.2) \quad |\kappa_A - \kappa_B| \leq 1.$$

We recall a well known factorization of the resolvent difference of  $A$  and  $B$  with the help of scalar functions which can be viewed as Weyl functions or  $Q$ -functions corresponding to  $A$  and  $B$ , respectively; cf. [13, 25]. The present version of Proposition 2.2 is taken from [3]. For the convenience of the reader a short proof of the resolvent formula with the help of boundary triples and their  $\gamma$ -fields and Weyl functions (see [12, 13, 14]) is given.

**Proposition 2.2.** *Let  $A$  and  $B$  be selfadjoint operators in the Krein space  $(\mathcal{H}, [\cdot, \cdot])$  which satisfy (2.1). Then there exist holomorphic functions  $M_A : \rho(A) \rightarrow \mathbb{C}$ ,  $M_B : \rho(B) \rightarrow \mathbb{C}$  symmetric with respect to the real line and vectors  $\varphi_A, \varphi_B$  in  $\mathcal{H}$  such that the following holds.*

(i) *For  $\gamma_A(\lambda) := (1 + (\lambda - \lambda_0)(A - \lambda)^{-1})\varphi_A$ ,  $\lambda \in \rho(A)$ , we have*

$$M_A(\lambda) - M_A(\bar{\omega}) = (\lambda - \bar{\omega})[\gamma_A(\lambda), \gamma_A(\omega)], \quad \lambda, \omega \in \rho(A).$$

(ii) *For  $\gamma_B(\lambda) := (1 + (\lambda - \lambda_0)(B - \lambda)^{-1})\varphi_B$ ,  $\lambda \in \rho(B)$ , we have*

$$M_B(\lambda) - M_B(\bar{\omega}) = (\lambda - \bar{\omega})[\gamma_B(\lambda), \gamma_B(\omega)], \quad \lambda, \omega \in \rho(B).$$

(iii) *For  $\lambda \in \rho(A) \cap \rho(B)$  we have  $M_B(\lambda) = -\frac{1}{M_A(\lambda)}$  and*

$$(A - \lambda)^{-1} - (B - \lambda)^{-1} = \frac{1}{M_A(\lambda)}[\cdot, \gamma_A(\bar{\lambda})]\gamma_A(\lambda) = -\frac{1}{M_B(\lambda)}[\cdot, \gamma_B(\bar{\lambda})]\gamma_B(\lambda).$$

*Proof.* Consider  $S = A \cap B$ , which is a (possibly nondensely defined) closed symmetric operator in  $(\mathcal{H}, [\cdot, \cdot])$  of defect one. As in [4, Corollary 2.5] it follows that there exists a boundary triple  $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$  for the adjoint  $S^+$  such that  $A = S^+ \upharpoonright \ker \Gamma_0$  and  $B = S^+ \upharpoonright \ker \Gamma_1$ . Let  $\gamma$  and  $M$  be the corresponding  $\gamma$ -field and Weyl function and define  $\varphi_A := \gamma(\lambda_0)$ . It follows from the property  $\gamma(\lambda) = (1 + (\lambda - \lambda_0)(A - \lambda)^{-1})\gamma(\lambda_0)$ ,  $\lambda \in \rho(A)$ , that  $\gamma_A = \gamma$  holds. Moreover,  $M_A := M$  satisfies the formula in (i). Observe that  $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$  is also a boundary triple for  $S^+$ . Let  $\tilde{\gamma}$  and  $\tilde{M}$  be the corresponding  $\gamma$ -field and Weyl function and define  $\varphi_B := \tilde{\gamma}(\lambda_0)$ . As above it follows that  $\gamma_B = \tilde{\gamma}$  and  $M_B := \tilde{M}$  satisfies the assertion in (ii). It follows from the definition of the Weyl function corresponding to a boundary triple that  $\tilde{M}(\lambda) = -M(\lambda)^{-1}$ , and hence  $M_B(\lambda) = -M_A(\lambda)^{-1}$ ,  $\lambda \in \rho(A) \cap \rho(B)$ , as stated in (iii). The formula in (iii) is a special case of [12, Theorem 2.1] (see also [14, Theorem 3.1]).  $\square$

From now on we will suppose that the following assumption is satisfied.

**Assumption (I).** Let  $A$  and  $B$  be selfadjoint operators in the Krein space  $(\mathcal{H}, [\cdot, \cdot])$  such that (2.1) holds for some (and hence for all)  $\lambda_0 \in \rho(A) \cap \rho(B)$ . Let  $I \subset \mathbb{R}$  be an open interval and assume that  $\rho(B) \cap I \neq \emptyset$  and that  $\sigma(A) \cap I$  consists only of isolated eigenvalues which are poles of the resolvent of  $A$ .

From Assumption (I) and general perturbation results (see e.g. [15, 17]) it follows that the set  $\sigma(B) \cap I$  consists only of eigenvalues which may only accumulate to the eigenvalues of infinite algebraic multiplicity of  $A$  or to the boundary of  $I$ . Note that any eigenvalue with an infinite dimensional root subspace of  $A$  is also an eigenvalue with an infinite dimensional root subspace of  $B$ . Furthermore, if  $\mu \in \rho(A) \cap I$  then either  $\mu \in \rho(B)$  or  $\mu \in \sigma_p(B)$  with  $\dim \ker(B - \mu) = 1$ .

The next proposition is essentially a consequence of Proposition 2.2. For a complete proof we refer the reader to [3, Corollary 2.2 and Lemma 2.4].

**Proposition 2.3.** *Let  $A, B$  be as in Assumption (I). Then for all  $\lambda \in I \cap \rho(A)$  we have*

- (i)  $\lambda \in \sigma_p(B)$  if and only if  $M_A(\lambda) = 0$ ;
- (ii)  $\lambda \in \sigma_{++}(B)$  if and only if  $M_A(\lambda) = 0$  and  $M'_A(\lambda) > 0$ ;
- (iii)  $\lambda \in \sigma_{--}(B)$  if and only if  $M_A(\lambda) = 0$  and  $M'_A(\lambda) < 0$ .

Let  $A$  and  $B$  be as in Assumption (I). According to [3, Lemma 2.5] the following holds: If  $\mu \in I \cap \sigma_{++}(A)$  ( $\mu \in I \cap \sigma_{--}(A)$ ) with  $\mu \in \rho(B)$  then the function  $M_A$  has a pole at  $\mu$  of order one with

$$\lim_{\lambda \nearrow \mu} M_A(\lambda) = +\infty, \quad \lim_{\lambda \searrow \mu} M_A(\lambda) = -\infty$$

$$\left( \lim_{\lambda \nearrow \mu} M_A(\lambda) = -\infty, \quad \lim_{\lambda \searrow \mu} M_A(\lambda) = +\infty, \text{ respectively} \right).$$

This observation immediately leads to the following interlacing property.

**Proposition 2.4.** *Let  $A, B$  be as in Assumption (I). Let  $\mu_1, \mu_2 \in \rho(B) \cap I$  be such that the interval  $(\mu_1, \mu_2)$  satisfies  $(\mu_1, \mu_2) \subset \rho(A)$ .*

- (i) *If  $\mu_1, \mu_2 \in \sigma_{++}(A)$  then there exists  $\mu \in (\mu_1, \mu_2)$  with  $\mu \in \sigma_p(B) \setminus \sigma_{--}(B)$ .*
- (ii) *If  $\mu_1, \mu_2 \in \sigma_{--}(A)$  then there exists  $\mu \in (\mu_1, \mu_2)$  with  $\mu \in \sigma_p(B) \setminus \sigma_{++}(B)$ .*

### 3. EIGENVALUE ESTIMATES

Let  $A$  and  $B$  be as in Assumption (I) and assume that  $A$  has finitely many negative squares  $\kappa_A$ . The next theorem provides estimates from below and above on the number of distinct eigenvalues of  $B$  in terms of the number of distinct eigenvalues of  $A$ . This is the first main result of this paper. In order to formulate it, we denote the numbers of distinct eigenvalues of  $A$  and  $B$  in  $I$  by  $n_A(I)$  and  $n_B(I)$ , respectively,

$$n_A(I) = \#\{\lambda : \lambda \in I \cap \sigma_p(A)\} \quad \text{and} \quad n_B(I) = \#\{\lambda : \lambda \in I \cap \sigma_p(B)\},$$

and we denote the number of common eigenvalues of  $A$  and  $B$  in  $I$  by  $n_{A,B}(I)$ ,

$$n_{A,B}(I) = \#\{\lambda : \lambda \in I \cap \sigma_p(A) \cap \sigma_p(B)\};$$

here the symbol  $\#$  stands for the number of elements in a given set. We emphasize that the multiplicities of the eigenvalues are not counted here. For a subset  $\Delta$  of  $I$  the numbers  $\mathfrak{K}_A(\Delta)$  and  $\mathfrak{K}_B(\Delta)$  are defined as

$$(3.1) \quad \mathfrak{K}_A(\Delta) = \#\{\lambda \in \sigma_p(A) \cap \Delta : \lambda \in (\mathbb{R}^+ \setminus \sigma_{++}(A)) \cup (\mathbb{R}^- \setminus \sigma_{--}(A))\}$$

and

$$\mathfrak{K}_B(\Delta) = \#\{\lambda \in \sigma_p(B) \cap \Delta : \lambda \in (\mathbb{R}^+ \setminus \sigma_{++}(B)) \cup (\mathbb{R}^- \setminus \sigma_{--}(B))\},$$

respectively; where  $\mathbb{R}^+ = (0, \infty)$  and  $\mathbb{R}^- = (-\infty, 0)$ . Both numbers  $\mathfrak{K}_A(\Delta)$  and  $\mathfrak{K}_B(\Delta)$  are finite by Theorem 2.1 and satisfy the estimates

$$(3.2) \quad \mathfrak{K}_A(\Delta) \leq \kappa_A \quad \text{and} \quad \mathfrak{K}_B(\Delta) \leq \kappa_B.$$

We are ready to formulate our first main result. We mention that all estimates in the next theorem will turn out to be sharp, see Theorem 4.1 below.

**Theorem 3.1.** *Let  $A, B$  be as in Assumption (I) and assume that  $A$  has  $\kappa_A$  negative squares. Then the following estimates hold.*

(i) If  $n_A(I) < \infty$  and  $0 \notin I$  then

$$n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + 1.$$

(ii) If  $n_A(I) < \infty$  and  $0 \in I$  then

$$n_B(I) \geq n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - \begin{cases} 3 & \text{if } 0 \in \rho(B) \cap c(A), \\ 2 & \text{if } 0 \in \rho(B) \setminus c(A), \\ 1 & \text{if } 0 \in \sigma(B), \end{cases}$$

and

$$n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + \begin{cases} 3 & \text{if } 0 \in \rho(A) \cap c(B), \\ 2 & \text{if } 0 \in \rho(A) \setminus c(B), \\ 1 & \text{if } 0 \in \sigma(A). \end{cases}$$

(iii) We have  $n_A(I) = \infty$  if and only if  $n_B(I) = \infty$ .

*Proof.* (i) Let  $n_A(I) < \infty$  and  $0 \notin I$ . It is no restriction to assume  $I \subset \mathbb{R}^+$ . Denote by  $n \geq 0$  the number of points in the open interval  $I$  which are in  $\sigma(A) \cap \sigma(B)$  or in  $\sigma(A) \setminus \sigma_{++}(A)$ . Obviously,

$$(3.3) \quad n = n_{A,B}(I) + \mathfrak{K}_A(I \cap \rho(B)).$$

The interval  $I$  without these  $n$  points decomposes into  $n+1$  open intervals  $I_j$ ,  $j = 1, \dots, n+1$ . Each  $I_j$  satisfies  $I_j \cap \sigma(A) \subset \sigma_{++}(A) \cap \rho(B)$ . In view of Proposition 2.4 we have for  $j = 1, \dots, n+1$

$$n_A(I_j) - 1 \leq n_B(I_j).$$

Summing up, we obtain

$$(3.4) \quad \left( \sum_{j=1}^{n+1} n_A(I_j) \right) - n - 1 \leq \sum_{j=1}^{n+1} n_B(I_j).$$

With (3.4) and with the identity  $n_A(I) = n + \sum_{j=1}^{n+1} n_A(I_j)$  we obtain

$$\begin{aligned} n_B(I) &= \left( \sum_{j=1}^{n+1} n_B(I_j) \right) + n_{A,B}(I) \geq \left( \sum_{j=1}^{n+1} n_A(I_j) \right) - n - 1 + n_{A,B}(I) \\ &= n_A(I) - 2n - 1 + n_{A,B}(I). \end{aligned}$$

Then (3.3) implies the lower estimate in (i). By interchanging the role of  $A$  and  $B$ , also the upper estimate in (i) is shown.

(ii) Let  $n_A(I) < \infty$  and  $0 \in I$ . We show the lower estimate in (ii); the upper estimate follows by interchanging the roles of  $A$  and  $B$ . We apply (i) to the intervals  $I \cap \mathbb{R}^+$  and  $I \cap \mathbb{R}^-$  and obtain

$$n_B(I \cap \mathbb{R}^\pm) \geq n_A(I \cap \mathbb{R}^\pm) - n_{A,B}(I \cap \mathbb{R}^\pm) - 2\mathfrak{K}_A(I \cap \mathbb{R}^\pm \cap \rho(B)) - 1.$$

By adding the inequality for the interval  $I \cap \mathbb{R}^-$  and the inequality for  $I \cap \mathbb{R}^+$  we derive

$$(3.5) \quad n_B(I \setminus \{0\}) \geq n_A(I \setminus \{0\}) - n_{A,B}(I \setminus \{0\}) - 2\mathfrak{K}_A((I \setminus \{0\}) \cap \rho(B)) - 2.$$

Obviously, if zero is in  $\rho(A)$ , then  $n_A(I) = n_A(I \setminus \{0\})$ . Otherwise  $n_A(I) = n_A(I \setminus \{0\}) + 1$  and the same identities are valid for  $A$  replaced by  $B$ . Similarly, if zero is in  $\sigma(A) \cap \sigma(B)$ , then  $n_{A,B}(I) = n_{A,B}(I \setminus \{0\}) + 1$ . Otherwise  $n_{A,B}(I) = n_{A,B}(I \setminus \{0\})$ . From the definition of

the number  $\mathfrak{K}_A$  in (3.1) it follows  $\mathfrak{K}_A(\Delta) = \mathfrak{K}_A(\Delta \setminus \{0\})$  for every subset  $\Delta$  of  $I$  and hence we have with (3.5)

$$(3.6) \quad n_B(I) \geq n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - \begin{cases} 3 & \text{if } 0 \in \rho(B) \cap \sigma(A), \\ 2 & \text{if } 0 \in \rho(B) \cap \rho(A), \\ 1 & \text{if } 0 \in \sigma(B). \end{cases}$$

In order to show the lower bound in (ii) it remains to consider the case  $0 \in (\rho(B) \cap \sigma(A)) \setminus c(A)$  and to show

$$(3.7) \quad n_B(I) \geq n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - 2.$$

For a spectral point  $\lambda$  of  $A$  which is in  $I \setminus c(A)$  we have  $\lambda \in \sigma_{++}(A) \cup \sigma_{--}(A)$ . Hence, if  $0 \in (\rho(B) \cap \sigma(A)) \setminus c(A)$ , then

$$0 \in \rho(B) \cap (\sigma_{++}(A) \cup \sigma_{--}(A))$$

and it is sufficient to show (3.7) for  $0 \in \rho(B) \cap \sigma_{++}(A)$ . The case  $0 \in \rho(B) \cap \sigma_{--}(A)$  is treated similarly. Therefore, we assume in the following

$$(3.8) \quad 0 \in \rho(B) \cap \sigma_{++}(A).$$

Then there exists  $\varepsilon > 0$  such that

$$[-\varepsilon, 0) \subset \rho(A) \cap \rho(B)$$

and we set

$$A_\varepsilon := A + \varepsilon, \quad B_\varepsilon := B + \varepsilon, \quad \text{and} \quad I_\varepsilon := \{\lambda + \varepsilon : \lambda \in I\}.$$

It is not difficult to see that Theorem 2.1 and (3.8) yield that  $A_\varepsilon$  ( $B_\varepsilon$ ) is also an operator with  $\kappa_A$  (resp.  $\kappa_B$ ) negative squares and that Assumption (I) holds for  $A$  and  $B$  replaced by  $A_\varepsilon$  and  $B_\varepsilon$ . Furthermore, the spectrum of  $A_\varepsilon$  in  $I_\varepsilon$  consists of isolated eigenvalues only. Then (3.6) is also valid for  $A_\varepsilon$ ,  $B_\varepsilon$ ,  $I_\varepsilon$  and, as  $0 \in \rho(A_\varepsilon) \cap \rho(B_\varepsilon)$ , we have

$$(3.9) \quad n_{B_\varepsilon}(I_\varepsilon) \geq n_{A_\varepsilon}(I_\varepsilon) - n_{A_\varepsilon, B_\varepsilon}(I_\varepsilon) - 2\mathfrak{K}_{A_\varepsilon}(I_\varepsilon \cap \rho(B_\varepsilon)) - 2.$$

Obviously, we have

$$(3.10) \quad n_A(I) = n_{A_\varepsilon}(I_\varepsilon), \quad n_B(I) = n_{B_\varepsilon}(I_\varepsilon), \quad \text{and} \quad n_{A,B}(I) = n_{A_\varepsilon, B_\varepsilon}(I_\varepsilon).$$

In particular, (3.8) implies

$$(3.11) \quad \mathfrak{K}_A(I \cap \rho(B)) = \mathfrak{K}_{A_\varepsilon}(I_\varepsilon \cap \rho(B_\varepsilon))$$

and, therefore, (3.7) follows from (3.9), (3.10), and (3.11).

(iii) Assume  $n_A(I) = \infty$ . Then there exists a sequence of subintervals  $(I_n)$  of  $I$  such that  $n_A(I_n)$  is finite but greater than  $n$ ,  $n \in \mathbb{N}$ . It follows from Theorem 2.1 and (3.2) that

$$\mathfrak{K}_A(I_n \cap \rho(B)) \leq \mathfrak{K}_A(I) \leq \kappa_A < \infty.$$

Then the inequalities proved in (i) and (ii) applied to  $I_n$  give

$$n_B(I_n) + n_{A,B}(I_n) \geq n_A(I_n) - 2\mathfrak{K}_A(I_n \cap \rho(B)) - 3 > n - 2\kappa_A - 3.$$

And, for  $n \rightarrow \infty$ , we conclude  $n_B(I) = \infty$ . Contrary, if  $n_B(I) = \infty$ , we conclude  $n_A(I) = \infty$  by a similar reasoning.  $\square$

In view of (2.2) and (3.2) the operator  $B$  has  $\kappa_B \leq \kappa_A + 1$  negatives squares and

$$\mathfrak{K}_A(I \cap \rho(B)) \leq \mathfrak{K}_A(I) \leq \kappa_A \quad \text{and} \quad \mathfrak{K}_B(I \cap \rho(A)) \leq \mathfrak{K}_B(I) \leq \kappa_B \leq \kappa_A + 1.$$

This implies the following corollary.

**Corollary 3.2.** *If  $n_A(I) < \infty$  and  $0 \notin I$  then*

$$n_A(I) - n_{A,B}(I) - 2\kappa_A - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\kappa_A + 3.$$

*If  $n_A(I) < \infty$  and  $0 \in I$  then*

$$n_A(I) - n_{A,B}(I) - 2\kappa_A - 3 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 2\kappa_A + 5.$$

#### 4. SHARPNESS

In this section we show, roughly speaking, that all the estimates in Theorem 3.1 are sharp. Here sharpness means the existence of an interval  $I$  and of two operators  $A, B$  satisfying Assumption (I) for which we obtain equality in the formulas in (i) and (ii) in Theorem 3.1. In particular, for given natural numbers  $k, n$  which stand for  $\mathfrak{K}_B(I \cap \rho(A))$  or  $\mathfrak{K}_A(I \cap \rho(B))$  and  $n_A(I)$ , respectively, and every number  $p (= n_{A,B}(I))$  smaller than  $n$  we find for each of the inequalities in (i) and (ii) of Theorem 3.1  $A, B$  and  $I$  such that equality holds. In this sense, the inequalities in Theorem 3.1 are optimal and next theorem complements the estimates in Theorem 3.1. Its proof relies on minimal realizations of rational generalized Nevanlinna functions in finite dimensional Pontryagin spaces, and makes use of boundary triple techniques in a similar way as in the proof of Proposition 2.2. A similar method was used in [3] to show sharpness of related eigenvalue estimates for nonnegative operators.

**Theorem 4.1.** (i) *For  $k, n \in \mathbb{N}$  and each number  $p \in \{0, 1, \dots, n\}$  there exists an open interval  $I \subset \mathbb{R}^+$ , a finite dimensional Krein space  $\mathcal{K}$  and selfadjoint matrices  $A$  and  $B$  in  $\mathcal{K}$  which satisfy Assumption (I) such that*

$$n_B(I) = n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + 1$$

*holds with  $n_A(I) = n$ ,  $\mathfrak{K}_B(I \cap \rho(A)) = k$  and  $n_{A,B}(I) = p$ .*

(ii) *Let  $k, n \in \mathbb{N}$  and  $p \in \{0, 1, \dots, n\}$  be numbers with  $n - p - 2k \geq 1$ . Then there exists an open interval  $I \subset \mathbb{R}^+$ , a finite dimensional Krein space  $\mathcal{K}$  and selfadjoint matrices  $A$  and  $B$  in  $\mathcal{K}$  which satisfy Assumption (I) such that*

$$n_B(I) = n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - 1$$

*holds with  $n_A(I) = n$ ,  $\mathfrak{K}_A(I \cap \rho(B)) = k$  and  $n_{A,B}(I) = p$ .*

(iii) *For  $k, n \in \mathbb{N}$  and each number  $p \in \{0, 1, \dots, n\}$  there exists an open interval  $I \subset \mathbb{R}$  with  $0 \in I$ , a finite dimensional Krein space  $\mathcal{K}$  and selfadjoint matrices  $A$  and  $B$  in  $\mathcal{K}$  which satisfy Assumption (I) such that*

$$n_B(I) = n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + \begin{cases} 3 & \text{if } 0 \in \rho(A) \cap c(B), \\ 2 & \text{if } 0 \in \rho(A) \setminus c(B), \\ 1 & \text{if } 0 \in \sigma(A) \end{cases}$$

*holds with  $n_A(I) = n$ ,  $\mathfrak{K}_B(I \cap \rho(A)) = k$  and  $n_{A,B}(I) = p$ .*

(iv) *Let  $k, n \in \mathbb{N}$  and  $p \in \{0, 1, \dots, n\}$  be numbers with the following property: If  $0 \in \sigma(B)$  then  $n - p - 2k \geq 1$ ; if  $0 \in \rho(B) \setminus c(A)$  then  $n - p - 2k \geq 2$ , and, if  $0 \in \rho(B) \cap c(A)$  then  $n - p - 2k \geq 3$ . Then there exists an open interval  $I \subset \mathbb{R}$  with  $0 \in I$ , a finite dimensional Krein space  $\mathcal{K}$  and selfadjoint matrices  $A$  and  $B$  in  $\mathcal{K}$  which satisfy Assumption (I) such that*

$$n_B(I) = n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A(I \cap \rho(B)) - \begin{cases} 3 & \text{if } 0 \in \rho(B) \cap c(A), \\ 2 & \text{if } 0 \in \rho(B) \setminus c(A), \\ 1 & \text{if } 0 \in \sigma(B) \end{cases}$$



holds with  $n_A(I) = n$ ,  $\mathfrak{K}_A(I \cap \rho(A)) = k$  and  $n_{A,B}(I) = p$ .

*Proof.* (i) Let  $k, n \in \mathbb{N}$  and fix positive numbers  $\lambda_0, \dots, \lambda_{n+1}$ ,  $\mu_1, \dots, \mu_{2k+1}$ , and  $v_1, \dots, v_n$  such that

$$0 < \lambda_0 < \mu_1 < \dots < \mu_{2k+1} < \lambda_1 < v_1 < \lambda_2 < \dots < v_n < \lambda_{n+1}$$

and set  $I := (\lambda_0, \lambda_{n+1})$ . The rational function

$$M(\lambda) := -\frac{(\lambda - \mu_1) \dots (\lambda - \mu_{2k+1})(\lambda - v_1) \dots (\lambda - v_n)}{(\lambda - \lambda_0) \dots (\lambda - \lambda_{n+1})(\lambda + 1)^{2k-1}}$$

has the following obvious properties:

- (a)  $M$  is symmetric with respect to the real axis,  $M(\bar{\lambda}) = \overline{M(\lambda)}$ ,
- (b)  $M$  has  $n$  simple poles  $\lambda_1, \dots, \lambda_n$  in  $I$ ,
- (c)  $M$  has  $2k+1$  simple zeros  $\mu_1, \dots, \mu_{2k+1}$  in  $(\lambda_0, \lambda_1) \subset I$  such that

$$M'(\mu_{2j+1}) > 0, \quad j = 0, \dots, k, \quad \text{and} \quad M'(\mu_{2j}) < 0, \quad j = 1, \dots, k,$$

and  $M$  has  $n$  simple zeros  $v_1, \dots, v_n \in I$  such that  $M'(v_l) > 0$ ,  $l = 1, \dots, n$ ,

- (d)  $\lim_{\lambda \rightarrow \pm\infty} M(\lambda) = -1$ .

Next we argue in the same way as in the proof of [3, Theorem 3.5] making use of a minimal realization of the function  $M$  as a Weyl function of some boundary triple; cf. [2, 16, 21]. More precisely, since  $M$  is a rational generalized Nevanlinna function there exists a Pontryagin space  $(\mathcal{H}, [\cdot, \cdot])$ , a (possibly nondensely defined) symmetric operator  $S$  with defect one and a boundary triple  $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$  for the adjoint  $S^+$  such that the corresponding Weyl function coincides with  $M$ , see [2, Corollary 3.5]. The model can be chosen minimal, in which case  $\mathcal{H}$  is a finite dimensional space and  $A := S^+ \upharpoonright \ker \Gamma_0$  is a selfadjoint matrix with eigenvalues located at the poles of  $M$ . In particular,  $A$  has no multivalued part as  $M$  has no pole at  $\pm\infty$ . Note that  $\sigma(A) \cap I$  consists of the  $n$  distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ , and hence

$$(4.1) \quad n_A(I) = n.$$

Next we use that  $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$  is a boundary triple for  $S^+$  with Weyl function  $-M^{-1}$  and that  $B := S^+ \upharpoonright \ker \Gamma_1$  is a selfadjoint matrix in  $\mathcal{H}$ . Note that  $B$  has no multivalued part as  $-M^{-1}$  has no pole at  $\pm\infty$ . Since both  $A$  and  $B$  are selfadjoint extensions of the symmetric (nondensely defined) operator  $S$  with defect one the difference of  $A$  and  $B$  is a rank one operator and, hence, the difference of their resolvents is a rank one operator. Therefore Assumption (I) is satisfied. Moreover, the zeros of  $M$  in  $I$  coincide with  $\sigma(B) \cap I$ . Hence  $B$  has  $2k+1$  eigenvalues in the interval  $(\lambda_0, \lambda_1)$ , where

$$\sigma_{++}(B) \cap (\lambda_0, \lambda_1) = \{\mu_1, \mu_3, \dots, \mu_{2k+1}\}, \quad \sigma_{--}(B) \cap (\lambda_0, \lambda_1) = \{\mu_2, \mu_4, \dots, \mu_{2k}\},$$

and one eigenvalue in each of the  $n = n_A(I)$  intervals  $(\lambda_1, \lambda_2), \dots, (\lambda_n, \lambda_{n+1})$ ; cf. Proposition 2.3 and (4.1). In particular, we have

$$n_B(I) = 2k+1 + n_A(I) \quad \text{and} \quad \mathfrak{K}_B(I \cap \rho(A)) = \mathfrak{K}_B((\lambda_0, \lambda_1)) = k,$$

and hence assertion (i) in the case  $n_{A,B}(I) = 0$  follows. In order to obtain the assertion in the remaining case  $1 \leq n_{A,B}(I) \leq n$  add orthogonally to  $A$  and  $B$  a matrix  $C$  with  $n_C(I) = n_{A,B}(I)$  distinct eigenvalues such that  $\sigma_p(C) \subset \sigma_p(A)$ . Then,

$$(4.2) \quad \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

have  $n_C(I)$  common eigenvalues in the interval  $I$ , and their resolvents differ by a rank one matrix. We have shown assertion (i). Observe that assertion (ii) follows by interchanging the roles of  $A$  and  $B$ .

(iii) Let  $k, n \in \mathbb{N}$ , fix  $\lambda_0, \dots, \lambda_{n+1}$ ,  $\mu_0, \dots, \mu_{2k+1}$ , and  $v_1, \dots, v_n$  such that

$$\lambda_0 < \mu_0 < 0 < \mu_1 < \dots < \mu_{2k+1} < \lambda_1 < v_1 < \lambda_2 < \dots < v_n < \lambda_{n+1}$$

and set  $I := (\lambda_0, \lambda_{n+1})$ . The rational function

$$N(\lambda) := -\frac{(\lambda - \mu_0)\lambda^2(\lambda - \mu_1)\dots(\lambda - \mu_{2k+1})(\lambda - v_1)\dots(\lambda - v_n)}{(\lambda - \lambda_0)\dots(\lambda - \lambda_{n+1})(\lambda - (\lambda_0 - 1))^{2k+2}}$$

has the following obvious properties:

- (a)  $N$  is symmetric with respect to the real axis,  $N(\bar{\lambda}) = \overline{N(\lambda)}$ ,
- (b)  $N$  has  $n$  simple poles  $\lambda_1, \dots, \lambda_n$  in  $I$ ,
- (c)  $N$  has a zero of multiplicity 2 at 0 and  $2k+2$  simple zeros  $\mu_0, \dots, \mu_{2k+1}$  in the interval  $(\lambda_0, \lambda_1) \subset I$  such that

$$(4.3) \quad N'(\mu_{2j+1}) > 0, \quad j = 0, \dots, k, \quad \text{and} \quad N'(\mu_{2j}) < 0, \quad j = 0, \dots, k,$$

and  $N$  has  $n$  simple zeros  $v_1, \dots, v_n \in I$  such that  $N'(v_l) > 0$ ,  $l = 1, \dots, n$ ,

$$(d) \quad \lim_{\lambda \rightarrow \pm\infty} N(\lambda) = -1.$$

Now use a minimal realization of  $N$  and show the existence of  $\mathcal{K}$  and  $A$  and  $B$  in exactly the same way as in the proof of assertion (i). Then  $\sigma(A) \cap I$  consists of the poles of  $N$  in  $I$ ,  $\sigma(A) \cap I = \{\lambda_1, \dots, \lambda_n\}$ . The zeros of  $N$  in  $I$  coincide with  $\sigma(B) \cap I$ . From (4.3) and Proposition 2.2 (ii) and (iii) we conclude

$$\mu_{2j+1} \in \sigma_{++}(B), \quad j = 0, \dots, k, \quad \text{and} \quad \mu_{2j} \in \sigma_{--}(B), \quad j = 0, \dots, k,$$

and we see, as  $\mu_0 < 0$  and  $0 < \mu_1 < \dots < \mu_{2k+1}$ ,

$$\mathfrak{K}_B(I \cap \rho(A)) = k.$$

Hence

$$n_B(I) = 2k + 3 + n = n_A(I) + 2\mathfrak{K}_B(I \cap \rho(A)) + 3.$$

As in (4.2) one treats the case  $1 \leq n_{A,B}(I) \leq n$ . The sharpness in the remaining two cases in (iii) can be shown analogously with the help of a minimal model for the generalized Nevanlinna function

$$P(\lambda) := -\frac{(\lambda - \mu_0)(\lambda - \mu_1)\dots(\lambda - \mu_{2k+1})(\lambda - v_1)\dots(\lambda - v_n)}{(\lambda - \lambda_0)\dots(\lambda - \lambda_{n+1})(\lambda - (\lambda_0 - 1))^{2k}},$$

which has the properties

- (a)  $P$  is symmetric with respect to the real axis,  $P(\bar{\lambda}) = \overline{P(\lambda)}$ ,
- (b)  $P$  has  $n$  simple poles  $\lambda_1, \dots, \lambda_n$  in  $I = (\lambda_0, \lambda_{n+1})$ ,
- (c)  $P$  has  $2k+2$  simple zeros  $\mu_0, \dots, \mu_{2k+1}$  in the interval  $(\lambda_0, \lambda_1) \subset I$  such that

$$P'(\mu_{2j+1}) > 0, \quad j = 0, \dots, k, \quad \text{and} \quad P'(\mu_{2j}) < 0, \quad j = 0, \dots, k,$$

and  $P$  has  $n$  simple zeros  $v_1, \dots, v_n \in I$  such that  $P'(v_l) > 0$ ,  $l = 1, \dots, n$ ,

$$(d) \quad \lim_{\lambda \rightarrow \pm\infty} P(\lambda) = -1,$$

and a minimal model for the generalized Nevanlinna function

$$Q(\lambda) := -\frac{(\lambda - \mu_0)(\lambda - \mu_1)\dots(\lambda - \mu_{2k+1})(\lambda - v_1)\dots(\lambda - v_n)}{(\lambda - \lambda_0)\lambda^2(\lambda - \lambda_1)\dots(\lambda - \lambda_{n+1})(\lambda - (\lambda_0 - 1))^{2k-2}},$$

which has the properties

- (a)  $Q$  is symmetric with respect to the real axis,  $Q(\bar{\lambda}) = \overline{Q(\lambda)}$ ,
- (b)  $Q$  has a pole of order 2 at 0 and  $n$  simple poles  $\lambda_1, \dots, \lambda_n$  in  $I = (\lambda_0, \lambda_{n+1})$ ,
- (c)  $Q$  has  $2k+2$  simple zeros  $\mu_0, \dots, \mu_{2k+1}$  in the interval  $(\lambda_0, \lambda_1) \subset I$  such that
$$Q'(\mu_{2j+1}) > 0, \quad j = 0, \dots, k, \quad \text{and} \quad Q'(\mu_{2j}) < 0, \quad j = 0, \dots, k,$$
and  $Q$  has  $n$  simple zeros  $v_1, \dots, v_n \in I$  such that  $Q'(v_l) > 0$ ,  $l = 1, \dots, n$ ,
- (d)  $\lim_{\lambda \rightarrow \pm\infty} Q(\lambda) = -1$ .

This completes the proof of assertion (iii). It is clear that (iv) follows by interchanging  $A$  and  $B$ .  $\square$

## 5. AN EXAMPLE: SINGULAR INDEFINITE STURM-LIOUVILLE PROBLEMS

In this section the general eigenvalue estimates are illustrated in a typical application from the theory of singular Sturm-Liouville problems with indefinite weight functions. We go beyond the so-called left-definite case, which was studied intensively from different points of view; cf. [6, 7, 8, 11, 18, 19, 20, 26].

Let  $r, p^{-1}, q \in L^1_{\text{loc}}(\mathbb{R})$  be real valued,  $p > 0$  and  $r \neq 0$  a.e., and consider the differential expression  $\tau$  on  $\mathbb{R}$ ,

$$\tau = \frac{1}{r} \left( -\frac{d}{dx} p \frac{d}{dx} + q \right).$$

We assume that  $\tau$  is in the limit point case at  $\pm\infty$  and that the weight function  $r$  has only one sign change at some point  $c \in \mathbb{R}$  such that  $r_+ = r \upharpoonright (c, \infty) > 0$  and  $r_- = r \upharpoonright (-\infty, c) < 0$  a.e.

The *indefinite* Sturm-Liouville operator  $B$  corresponding to  $\tau$  is defined by

$$Bf = \tau(f) = \frac{1}{r} ((-pf')' + qf), \quad f \in \text{dom } B,$$

where the domain  $\text{dom } B$  consists of all locally absolutely continuous functions  $f \in L^2(\mathbb{R}, |r|)$  such that  $pf'$  is locally absolutely continuous and  $\tau(f) \in L^2(\mathbb{R}, |r|)$ . Note that  $B$  is selfadjoint in the Krein space  $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$ , where  $[\cdot, \cdot]$  is given by

$$(5.1) \quad [f, g] = \int_{\mathbb{R}} f(x) \overline{g(x)} r(x) dx, \quad f, g \in L^2(\mathbb{R}, |r|).$$

Here  $L^2(\mathbb{R}, |r|)$  denotes the space of all equivalence classes of complex valued measurable functions  $f$  on  $\mathbb{R}$  such that

$$(5.2) \quad (f, f) = \int_{\mathbb{R}} |f(x)|^2 |r(x)| dx < \infty.$$

In the following we will also make use of the selfadjoint realizations  $T_+$  and  $T_-$  of  $\tau$  restricted to  $(c, \infty)$  and  $(-\infty, c)$ , respectively, with Dirichlet boundary conditions at  $c$  in the Hilbert spaces  $L^2((c, \infty), |r_+|)$  and  $L^2((-\infty, c), |r_-|)$ , respectively. Here  $L^2((c, \infty), |r_+|)$  and  $L^2((-\infty, c), |r_-|)$  stands for the spaces of all equivalence classes of square integrable complex valued functions  $f$  with  $\int_{(c, \infty)} |f|^2 |r_+| < \infty$  (resp.  $\int_{(-\infty, c)} |f|^2 |r_-| < \infty$ ). The domain of  $T_+$  and  $T_-$  consist of all locally absolutely continuous functions  $f$  in  $L^2((c, \infty), |r_+|)$  ( $L^2((-\infty, c), |r_-|)$ , respectively) which are zero in  $c$  such that  $pf'$  is locally absolutely continuous and the restrictions of  $\tau(f)$  are in  $L^2((c, \infty), |r_+|)$  and  $L^2((-\infty, c), |r_-|)$ , respectively.

Obviously, the direct sum  $T_+ \oplus T_-$  is closely related to the operator  $B$ , see also Proposition 5.1 below. As we are interested in operators with finitely many squares we will impose assumptions on  $T_+$  and  $T_-$  which implies that  $T_+ \oplus T_-$  and the indefinite Sturm-Liouville operator  $B$  have finitely many negative squares.

**Proposition 5.1.** *Assume that the essential spectrum of  $T_+$  and  $T_-$  satisfies*

$$(5.3) \quad \eta_+ := \min \sigma_{\text{ess}}(T_+) > 0 \quad \text{and} \quad \eta_- := \max \sigma_{\text{ess}}(T_-) < 0$$

*and that  $T_+$  has only finitely many eigenvalues in  $(-\infty, \eta_+)$  and  $T_-$  has only finitely many eigenvalues in  $(\eta_-, \infty)$ . Then the following holds.*

- (i) *The orthogonal sum  $T_+ \oplus T_-$  is a selfadjoint operator in the Hilbert space  $L^2(\mathbb{R}, |r|)$ .*
- (ii) *The indefinite Sturm-Liouville operator  $B$  has  $\kappa_B$  negative squares in the Krein space  $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$  with  $\rho(B) \neq \emptyset$  and*

$$\kappa_B \leq n_{T_+}((-\infty, 0)) + n_{T_-}((0, \infty)) + 1.$$

- (iii) *We have*

$$(5.4) \quad \dim \text{ran}((B - \lambda)^{-1} - ((T_+ \oplus T_-) - \lambda)^{-1}) = 1, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

*and the essential spectrum of  $T_+ \oplus T_-$  coincides with the essential spectrum of  $B$ . In particular,*

$$(\eta_-, \eta_+) \cap \sigma_{\text{ess}}(B) = \emptyset.$$

*Proof.* The selfadjointness of  $T_+$  and  $T_-$  implies (i). But the operator  $T_+ \oplus T_-$  is also selfadjoint in the Krein space  $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$ . Then [11, Remark 1.3] applied to  $T_+ \oplus T_-$  and the symmetric operator  $B \cap (T_+ \oplus T_-)$  gives  $\rho(B) \neq \emptyset$ . In order to show the assertion on  $\kappa_B$  in (ii), we consider the (definite) differential expression on  $\mathbb{R}$

$$\ell = \frac{1}{|r|} \left( -\frac{d}{dx} p \frac{d}{dx} + q \right).$$

Then  $\ell$  gives rise to a selfadjoint operator

$$(5.5) \quad Tf = \ell(f) = \frac{1}{|r|} ((-pf')' + qf) \quad f \in \text{dom } T,$$

in the Hilbert space  $L^2(\mathbb{R}, |r|)$ , where the domain  $\text{dom } T$  coincides with the domain of  $B$ . In view of the definition of the inner products in (5.1) and (5.2), we easily see

$$[Bf, f] = (Tf, f) \quad \text{for } f \in \text{dom } B = \text{dom } T.$$

Due to the limit point behaviour of  $\tau$  at  $\pm\infty$  all eigenvalues of  $T$  are simple. Hence the number of negative squares of the form  $[B\cdot, \cdot]$  equals the number of negative eigenvalues of the selfadjoint operator  $T$ ,  $\kappa_B = n_T((-\infty, 0))$ . The orthogonal sum of  $T_+$  with  $-T_-$  differs from  $T$  only in the boundary condition at  $c$ . Therefore

$$\dim(\text{ran}(T - \lambda)^{-1} - ((T_+ \oplus (-T_-)) - \lambda)^{-1}) = 1, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

This together with well-known perturbation results for selfadjoint operators in Hilbert spaces (see, e.g., [9, §9.3, Theorem 3]) gives

$$\kappa_B = n_T((-\infty, 0)) \leq n_{T_+ \oplus (-T_-)}((-\infty, 0)) + 1 = n_{T_+}((-\infty, 0)) + n_{T_-}((0, \infty)) + 1.$$

This shows (ii). The orthogonal sum of the operators  $T_+$  with  $T_-$  differs from  $B$  only in the boundary condition at  $c$  and (5.4) follows. The remaining assertions in (iii) are obvious.  $\square$

In Theorem 5.2 below we express the number of eigenvalues of  $B$  in a gap around zero in terms of the number of eigenvalues of the operators  $T_-$  and  $T_+$ . Observe that  $B$  is a selfadjoint operator in a Krein space and  $B$ , in general, is not left-definite. For such operators there are no results for the number of eigenvalues of  $B$  in terms of the coefficients  $r, p, q$  available. Contrary, as  $T_-$  and  $T_+$  are selfadjoint operators in Hilbert spaces and correspond to definite Sturm-Liouville expressions on the intervals  $(-\infty, c)$  and

$(c, \infty)$ , respectively, there are many results concerning the point spectrum. The estimates in the next theorem and Remark 5.4 below allow to translate such results to the indefinite case.

**Theorem 5.2.** *Assume (5.3) holds and that  $T_+$  has only finitely many eigenvalues in  $(-\infty, \eta_+)$  and  $T_-$  has only finitely many eigenvalues in  $(\eta_-, \infty)$ . Moreover, we assume*

$$(5.6) \quad \sigma_p(T_+) \cap \sigma_p(T_-) \cap (\eta_-, \eta_+) = \emptyset.$$

*Then the following holds.*

(i)  $n_B((\eta_-, \eta_+))$  is finite.

(ii) If  $0 \in \rho(B) \cap \rho(T_+) \cap \rho(T_-)$  then

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - n_{T_+}(\eta_-, 0) - n_{T_-}(0, \eta_+) - 2.$$

(iii) If  $0 \notin \rho(B) \cap \rho(T_+) \cap \rho(T_-)$  then

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - n_{T_+}(\eta_-, 0) - n_{T_-}(0, \eta_+) - 1.$$

*Proof.* As  $\ell$  is in the limit point case at  $\pm\infty$ , all eigenvalues of  $T_+$  and of  $T_-$  are simple. Assume  $\lambda \in \sigma_p(T_+) \cap \sigma_p(B)$ . Then the corresponding eigenfunction of the operator  $B$  is defined on  $\mathbb{R}$  and coincides on  $(c, \infty)$  with the eigenfunction of  $T_+$  corresponding to  $\lambda$ . Hence, it is zero at  $c$ . But this implies  $\lambda \in \sigma_p(T_-)$ , a contradiction to (5.6). The preceding argumentation remains valid if  $T_+$  interchanges with  $T_-$  and, hence, we obtain

$$(5.7) \quad n_{T_+ \oplus T_-, B}((\eta_-, \eta_+)) = 0.$$

The operator  $T_+ \oplus T_-$  is selfadjoint in the Krein space  $(L^2(\mathbb{R}, |r|), [\cdot, \cdot])$  and the number  $\mathfrak{K}_{T_+ \oplus T_-}((\eta_-, \eta_+) \cap \rho(B))$  defined in (3.1) can be calculated explicitly in terms of the eigenvalues of  $T_+$  and of  $T_-$ ,

$$(5.8) \quad \mathfrak{K}_{T_+ \oplus T_-}((\eta_-, \eta_+) \cap \rho(B)) = n_{T_+}(\eta_-, 0) + n_{T_-}(0, \eta_+).$$

By Proposition 5.1  $T_+ \oplus T_-$  and  $B$  satisfy Assumptions (I) and, hence, Theorem 3.1 shows

(i). Furthermore, Theorem 3.1, (5.7), and (5.8) give

$$n_B((\eta_-, \eta_+)) \geq n_{T_+ \oplus T_-}((\eta_-, \eta_+)) - 2n_{T_+}(\eta_-, 0) - 2n_{T_-}(0, \eta_+) - \begin{cases} 3 & \text{if } 0 \in \rho(B) \cap c(T_+ \oplus T_-), \\ 2 & \text{if } 0 \in \rho(B) \setminus c(T_+ \oplus T_-), \\ 1 & \text{if } 0 \in \sigma(B). \end{cases}$$

Note that  $0 \in c(T_+ \oplus T_-)$  if and only if  $0 \in \sigma_p(T_+) \cap \sigma_p(T_-)$ . By (5.6) the intersection of the set  $\sigma_p(T_+) \cap \sigma_p(T_-)$  with  $(\eta_-, \eta_+)$  is empty and hence the first case in the above inequality is not present. Moreover,

$$\begin{aligned} n_{T_+ \oplus T_-}((\eta_-, \eta_+)) &= n_{T_+}((\eta_-, \eta_+)) + n_{T_-}((\eta_-, \eta_+)) \\ &= n_{T_+}((0, \eta_+)) + n_{T_+}(\{0\}) + n_{T_+}((\eta_-, 0)) + n_{T_-}((0, \eta_+)) + n_{T_-}(\{0\}) + n_{T_-}((\eta_-, 0)) \end{aligned}$$

and we obtain the estimate

$$\begin{aligned} n_B((\eta_-, \eta_+)) &\geq n_{T_+}((0, \eta_+)) - n_{T_+}((\eta_-, 0)) - n_{T_-}((0, \eta_+)) + n_{T_-}((\eta_-, 0)) \\ &\quad + n_{T_+}(\{0\}) + n_{T_-}(\{0\}) - \begin{cases} 2 & \text{if } 0 \in \rho(B), \\ 1 & \text{if } 0 \in \sigma(B). \end{cases} \end{aligned}$$

If  $0 \in \rho(B) \cap \rho(T_+) \cap \rho(T_-)$  then  $n_{T_+}(\{0\}) = n_{T_-}(\{0\}) = 0$  and (ii) follows. If  $0 \in \sigma_p(B)$  then (5.7) gives  $0 \in \rho(T_+) \cap \rho(T_-)$  and  $n_{T_+}(\{0\}) = n_{T_-}(\{0\}) = 0$  implies (iii). If  $0 \in \rho(B)$  and  $0 \in \sigma_p(T_+)$ , then (5.6) implies  $0 \in \rho(T_-)$ . Hence  $n_{T_+}(\{0\}) = 1$ ,  $n_{T_-}(\{0\}) = 0$  and (iii) follows. The case  $0 \in \rho(B)$  and  $0 \in \sigma_p(T_-)$  is shown analogously.  $\square$

**Corollary 5.3.** *Assume in addition to the assumption in Theorem 5.2 that  $T_+$  is a nonnegative operator in the Hilbert space  $L^2((c, \infty), |r_+|)$  and  $T_-$  is a nonpositive operator in the Hilbert space  $L^2((-\infty, c), |r_-|)$ . Then the following holds.*

(i) *If  $0 \in \rho(B) \cap \rho(T_+) \cap \rho(T_-)$  then*

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - 2.$$

(ii) *If  $0 \notin \rho(B) \cap \rho(T_+) \cap \rho(T_-)$  then*

$$n_B((\eta_-, \eta_+)) \geq n_{T_+}(0, \eta_+) + n_{T_-}(\eta_-, 0) - 1.$$

In the situation of Corollary 5.3 we refer to [3] for a related estimate. In [3] the number of eigenvalues in a gap of the essential spectrum of  $B$  is estimated with the help of the number of eigenvalues of the definite Sturm-Liouville operator  $T$  (see (5.5)).

**Remark 5.4.** In Theorem 5.2 there are only estimates for the number  $n_B((\eta_-, \eta_+))$  from below. For the corresponding estimates from above in Theorem 3.1, applied to the operators  $T_+ \oplus T_-$  and  $B$ , the quantity  $\mathfrak{K}_B((\eta_-, \eta_+) \cap \rho(B))$  appears. In general it is difficult to find an estimate for this quantity in terms of the number of eigenvalues of  $T_+$  and  $T_-$  in the interval  $(\eta_-, \eta_+)$ . However one can use the general (and rough) estimate from Corollary 3.2 together with (5.7) and Proposition 5.1 (ii)

$$n_B((\eta_-, \eta_+)) \leq n_{T_+ \oplus T_-}((\eta_-, \eta_+)) + 2n_{T_+}((-\infty, 0)) + 2n_{T_-}((0, \infty)) + 5.$$

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